ON THE SPECTRUM OF MAGNETIC FIELD AND SCALAR IMPURITY FLUCTUATIONS IN ACOUSTIC TURBULENCE

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The fluctuations of a magnetic field in acoustic turbulence are examined. An equation is derived for the spectral tensor of homogeneous magnetic field fluctuations. In a certain limit case the spectrum of steady-state pulsations is obtained in the presence of an external source. It is shown that three kinds of spectra exist in an inertial subdomain, each of which corresponds to a definite domain in wave space. Analogous results have been obtained for the fluctuations of a homogeneous scalar impurity.

Under certain conditions, weak magnetic field fluctuations will grow with time in the presence of acoustic turbulence [1]. If

$$S = M^3 R_m \gg 1$$

where M is the Mach number and R_m the magnetic Reynolds number, then such an instability relative to the magnetic field will actually occur. Magnetic field fluctuations will grow until the Lorentz force ceases to influence the motion substantially, i.e., when the magnetic pressure is comparable to the plasma pressure. For such a case it is difficult to obtain the spectrum of steady-state fluctuations because there is no suitable small parameter here.

If $S \ll 1$, the fluctuations will damp out in the absence of external sources. The case is considered herein when there is an external source.

This is apparently the single possibility of prolonged coexistence of a magnetic field and acoustic turbulence since for $S \gg 1$ turbulence ceases to be acoustic in the long run. The problem of the spectrum of scalar impurity fluctuations in acoustic turbulence directly adjoins this problem because of the similarity of the equations for the fluctuations in both problems. Hence, this question will also be examined below.

1. Spectrum of Magnetic Field Fluctuations

1. It is simplest to assign the source as follows: Let us assume that there is a homogeneous magnetic field H_0, and the fluctuations are $h \ll H_0$.

There are two possibilities: $R_m \ll 1$, $R_m \gg 1$. G. S. Golitsyn [2] considered the case $R_m \ll 1$. Let $R_m \gg 1$. The equation for the magnetic field

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{rot}[\mathbf{v}, \mathbf{H}] + \nu_m \Delta \mathbf{H} \tag{1.1}$$

can be simplified because of the smallness of the fluctuations

$$\frac{\partial \mathbf{h}}{\partial t} = \operatorname{rot}[\mathbf{v}, \mathbf{H}_0] + \nu_m \Delta \mathbf{h} \tag{1.2}$$

or in a Fourier representation

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$$\frac{\partial \mathbf{h}}{\partial t} = i\left[\mathbf{k}\left[\frac{\mathbf{k}}{k}\varphi(\mathbf{k}), \mathbf{H}_0\right]\right] - \nu_m k^2 \mathbf{h}$$

Here ν_m is the magnetic viscosity, and $\varphi(k)/k$ is the Fourier transform of the hydrodynamic potential. Let us express $\mathbf{h}(k, t)$ in terms of the initial field

$$\mathbf{h}(\mathbf{k}, t) = \mathbf{h}(\mathbf{k}, 0)\, e^{-\nu_m k^2 t} + i\int_0^t e^{-\nu_m k^2 (t-t_1)}\left[\mathbf{k}\left[\frac{\mathbf{k}}{k}\varphi(\mathbf{k}, t_1), \mathbf{H}_0\right]\right] dt_1 \tag{1.3}$$

The time of variation of $\varphi(k, t)$ is $(kc)^{-1}$, where c is the speed of sound.

Furthermore, let us assume that the velocity and magnetic field fluctuations are homogeneous, and moreover, the velocity fluctuations are isotropic:

$$\begin{aligned} \langle \varphi(\mathbf{k}, t)\, \varphi^*(\mathbf{k}', t')\rangle &= \delta(\mathbf{k} - \mathbf{k}')\, f(k, |t - t'|) \\ \langle h_i(\mathbf{k}, t)\, h_j^*(\mathbf{k}', t)\rangle &= \delta(\mathbf{k} - \mathbf{k}')\, T_{ij}(k, H_0, t) \end{aligned} \tag{1.4}$$

Let us multiply (1.3) by the complex conjugate and average the right and left sides by assuming that $\mathbf{h}(k, 0)$ is statistically independent of $\varphi(k, t)$, whereupon we obtain

$$\begin{aligned} T_{ij}(\mathbf{k}, \mathbf{H}_0, t) &= T_{ij}(\mathbf{k}, \mathbf{H}_0, 0)\, e^{-2\nu_m k^2 t} \\ &+ \left\{\frac{k_i k_j}{k^2}(\mathbf{k}\mathbf{H}_0)^2 + H_{0i}H_{0j}k^2 - (\mathbf{k}\mathbf{H}_0)\left[H_{0i}k_j + H_{0j}k_i\right]\right\} \\ &\times \int_0^t\int_0^t \exp\left[-2\nu_m k^2 t + \nu_m k^2 (t_1 + t_2)\right] f(k, |t_1 - t_2|)\, dt_1 dt_2 \end{aligned} \tag{1.5}$$

To evaluate the integral in the right side of (1.5), let us perform still another Fourier transformation of $f(k, t)$ with respect to the time

$$f(k, t) = \int_{-\infty}^{+\infty} e^{i\omega t} J(k, \omega)\, d\omega \tag{1.6}$$

Furthermore

$$\int_0^t\int_0^t \exp\left[-2\nu_m k^2 t + \nu_m k^2 (t_1 + t_2)\right] f(k, |t_1 - t_2|)\, dt_1 dt_2 = \left(1 + e^{-2\nu_m k^2 t}\right)\int_{-\infty}^{+\infty} \frac{J(k, \omega)}{\nu_m^2 k^4 + \omega^2}\, d\omega - 2e^{-\nu_m k^2 t}\int_{-\infty}^{+\infty} \frac{e^{i\omega t} J(k, \omega)}{\nu_m^2 k^4 + \omega^2}\, d\omega \tag{1.7}$$

2. Because the initial field is considered uncorrelated with the velocities, it is meaningful to examine (1.5) for $t > \tau$, where τ is the correlation time, i.e., the time during which the system succeeds in "forgetting" the initial conditions. Since $R_m \gg 1$, the single parameter governing τ will be the correlation time in k-space, or the time of phonon interaction. Let us take this τ from [3]

$$\begin{aligned} \tau &= \frac{c}{E(k)\, k^2} \\ E(k) &= A v^2 \lambda^{-1/2} k^{-3/2} \qquad (A \approx 1) \end{aligned} \tag{1.8}$$

Here E(k) is the power spectrum of the acoustic oscillations, λ is the external scale, v^2 is the mean square amplitude; for $k > 1/\lambda$ the spectral domain corresponds to the inertial. Precisely in this domain does E(k) have the form (1.8), while for $k < 1/\lambda$ $E(k) \to 0$.

Let us first examine (1.7) in that domain of wave space where the relation

$$\tau < \frac{1}{\nu_m k^2} \quad \text{or} \quad k^{3/2} < \frac{v^2\lambda^{-1/2}}{c\nu_m} = k_1^{3/2} \tag{1.9}$$

is satisfied.

On the other hand, let $k > 1/\lambda$ (we will be interested in the inertial subdomain). The inequality

$$MR_m \gg 1 \tag{1.10}$$

should be satisfied for the domain of interest $1/\lambda \ll k \ll k_1$ to exist.

Now, let us consider (1.7). Let us note that for $t \gg \tau$

$$\int_{-\infty}^{+\infty} \frac{J(k,\omega)\, e^{i\omega t}}{\nu_m^2 k^4 + \omega^2}\, d\omega \to \frac{\pi J(k,0)}{\nu_m k^2}\, e^{-\nu_m k^2 t}$$

An equation for T_{ij} can now be composed:

$$\frac{\partial T_{ij}}{\partial t} + 2\nu_m k^2 T_{ij} = 2\nu_m k^2 \sigma_{ij}(\mathbf{k}, \mathbf{H}_0) \int_{-\infty}^{+\infty} \frac{J(k,\omega)}{\nu_m^2 k^4 + \omega^2}\, d\omega \tag{1.11}$$

$$\sigma_{ij} = \frac{k_i k_j (\mathbf{kH}_0)^2}{k^2} + H_{0i} H_{0j} k^2 - (\mathbf{kH}_0)\,[H_{0i} k_j + H_{0j} k_i]$$

For $t \to \infty$

$$T_{ij} \to \sigma_{ij} \int_{-\infty}^{+\infty} \frac{J(k,\omega)}{\nu_m^2 k^4 + \omega^2}\, d\omega \tag{1.12}$$

To evaluate (1.12) approximately (the exact value of $J(k, \omega)$ is unknown), let us note that

$$\frac{1}{\nu_m^2 k^4 + \omega^2} \approx \frac{\pi}{\nu_m k^2}\, \delta(\omega)$$

Moreover, $J(k, \omega)$ has a sharp maximum at $\omega = \pm ck$. Hence, we represent $J(k, \omega)$ as follows:

$$J(k,\ \omega) = J(k,\ 0)\, e(ck - \omega)\, e(ck + \omega) + {}^1\!/_2 f(k,\ 0)\, [\delta(\omega + ck) + \delta(\omega - ck)] \tag{1.13}$$

where $e(x)$ is a unit step: $e(x) = 1$ for $x > 0$ and $e(x) = 0$ for $x \leq 0$. Now (1.12) is evaluated as

$$\int_{-\infty}^{+\infty} \frac{J(k,\omega)}{\nu_m^2 k^4 + \omega^2}\, d\omega \approx \frac{\pi J(k,0)}{\nu_m k^2} + \frac{f(k,0)}{c^2 k^2} \qquad \left(f(k,0) = \frac{E(k)}{4\pi k^2}\right) \tag{1.14}$$

where $J(k, 0)$ has been obtained in [1]

$$J(k,0) = \frac{\pi k}{2c^3} \int_{1/_2 k}^{\infty} f^2(q,0)\, dq \tag{1.15}$$

If $E(k)$ has the form (1.8), then for $k > 1/\lambda$

$$J(k,0) = \frac{A^2 v^4}{3\pi c^3 \lambda} k^{-5} \tag{1.16}$$

Since

$$\frac{\pi J(k,0)}{\nu_m k^2} \gg \frac{f(k,0)}{c^2 k^2}$$

in the domain $1/\lambda < k < k_1$, the stationary spectral tensor T_{ij} has the form

$$T_{ij} = {}^1\!/_3 \sigma_{ij} \nu_m^{-1} v^4 c^{-3} \lambda^{-1} k^{-7} \tag{1.17}$$

It is seen from (1.17) and (1.11) that the spectrum is essentially anisotropic. The trace of the tensor is

$$\sigma_{ii} = H_0^2 k^2 - (\mathbf{H}_0 \mathbf{k})^2$$

Integrating T_{ij} with respect to the solid angle and multiplying by k^2, we obtain the power spectrum of the magnetic pulsations

$$F(k) = {}^8/_9 A^2 H_0^2 \pi \nu_m^{-1} v^4 c^{-3} \lambda^{-1} k^{-3} \tag{1.18}$$

Hence, it is easy to evaluate the energy of the magnetic pulsations

$$\frac{h^2}{8\pi} = \frac{1}{8\pi} \int_{1/\lambda}^{\infty} F(k)\, dk = H_0^2 \frac{S}{18}$$

Since $S \ll 1$, then $h \ll H_0$.

3. Now, let

$$\tau > 1/\nu_m k^2$$

i.e., $k > k_1$, but $k < c\nu_m^{-1}$. In this domain of the wave space the correlation between the magnetic field and the velocity field is established in the time $\tau_1 = (\nu_m k^2)^{-1}$ so that it is necessary to examine (1.7) for $t > \tau_1$. Taking this circumstance into account, we obtain the stationary spectral tensor T_{ij} and $F(k)$

$$T_{ij} = \sigma_{ij} \int_{-\infty}^{+\infty} \frac{J(k, \omega)}{\nu_m^2 k^4 + \omega^2} d\omega \approx \sigma_{ij} \frac{f(k, 0)}{c^2 k^2} \tag{1.19}$$

$$F(k) = \frac{2}{3} H_0^2 \frac{E(k)}{c^2}$$

And, finally, for $k > c\nu_m^{-1} = k_2$

$$T_{ij} \approx \sigma_{ij} \frac{f(k, 0)}{\nu_m^2 k^4}, \qquad F(k) = \frac{2}{3} H_0^2 \frac{E(k)}{\nu_m^2 k^2} \tag{1.20}$$

This latter spectrum is the G. S. Golitsyn spectrum [2].

Therefore, in the case under consideration, when $R_m \ll M^{-3}$, there are the following possibilities:

1) $R_m \ll M$; then $k_2 \ll 1/\lambda$ and for $k > 1/\lambda$ the spectrum (1.20) is established;

2) $M \ll R_m \ll M^{-1}$; then $k_2 \gg 1/\lambda$ but $k_1 \ll 1/\lambda$. Therefore, for $1/\lambda < k < k_2$ the function $F(k)$ corresponds to (1.19), while for $k > k_2$ the function $F(k)$ corresponds to (1.20);

3) $M^{-1} \ll R_m \ll M^{-3}$; then $k_2 \gg k_1 \gg 1/\lambda$. Therefore, for $1/\lambda < k < k_1$ the spectrum (1.18) is established, for $k_1 < k < k_2$ the spectrum (1.19), and for $k > k_2$ the spectrum (1.20).

Let us turn attention to the following circumstance: The spectra (1.19) and (1.20) will exist even if there is no interaction between the acoustic oscillations, if they are a set of random noises. Hence (1.19) is due to the magnetic field "tracking" the oscillations, as actually occurs since the period of the oscillations in this spectral domain is considerably less than the time to damp of the field, and (1.20) represents the fluctuations whose generation (due to the velocities) is canceled by the damping. Just the spectrum (1.18) originates in the presence of interaction between the oscillations (i.e., turbulence, in substance) and the particles behind which the magnetic field "follows."

2. Spectrum of Scalar Impurity Fluctuations

In this problem let us start from the equation

$$\frac{\partial f}{\partial t} + \operatorname{div} \mathbf{v} f = \mu \Delta f \tag{2.1}$$

Here f is the density of the scalar impurity (temperature, say); μ is the coefficient of molecular temperature conduction (if it is temperature) or an analogous transport coefficient. The role of R_m is played by the Peclet number P in this theory. Let us assume that $P \gg 1$, but

$$S_\mu = M^3 P \ll 1 \tag{2.2}$$

Condition (2.2) is necessary for the fluctuations to be small

$$f = f_0 + f_1, \quad f_0 = \text{const}, \quad f_1 \ll f_0$$

(f_1 are the fluctuations)

The foundation of this assertion will be given below. Using the smallness of f_1, let us simplify (2.1)

$$\frac{\partial f_1}{\partial t} + f_0 \operatorname{div} \mathbf{v} = \mu \Delta f_1 \tag{2.3}$$

or in the Fourier representation

$$\frac{\partial f_1(\mathbf{k}, t)}{\partial t} + \mu k^2 f(\mathbf{k}, t) = -i f_0 k \varphi(\mathbf{k}, t) \tag{2.4}$$

$$f_1(\mathbf{k}, t) = f(\mathbf{k}, 0) e^{-\mu k^2 t} - i f_0 k \int_0^t e^{-\mu k^2 (t - t_1)} \varphi(\mathbf{k}, t_1) \, dt_1 \tag{2.5}$$

Furthermore, assuming

$$\langle f_1(\mathbf{k}, t) f_1^*(\mathbf{k}', t) \rangle = \delta(\mathbf{k} - \mathbf{k}') T(k, t) \tag{2.6}$$

we obtain an equation for T by using the same method as for the magnetic fields:

$$\frac{\partial T(k, t)}{\partial t} + 2\mu k^2 T(k, t) = 2\mu k^4 f_0^2 \int_{-\infty}^{+\infty} \frac{J(k, \omega)}{\mu^2 k^4 + \omega^2} d\omega \tag{2.7}$$

As $t \to \infty$ the following stationary spectrum is built up:

$$T \to k^2 f_0^2 \int_{-\infty}^{+\infty} \frac{J(k, \omega)}{\mu^2 k^4 + \omega^2} d\omega \tag{2.8}$$

Upon compliance with the condition $MP \gg 1$, three sections of the spectrum are built up exactly as for the magnetic fluctuations. The sole difference is that the spectral function T is independent of the angle in this case. This is natural: There is no isolated direction in this problem. Here

$$k_1^{3/2} = \frac{v^2 \lambda^{-1/2}}{c\mu}, \quad k_2 = c\mu^{-1}$$

For $1/\lambda < k < k_1$ the power spectrum H(k) is similar to (1.18), for $k_1 < k < k_2$ it is similar to (1.19) and for $k > k_2$ to (1.20).

The root-mean-square value of the pulsations is

$$\langle f_1^2 \rangle \approx f_0^2 S_\mu$$

Thus, the assertion presented above about the smallness of the pulsations upon compliance with (2.2) is supported.

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